Faculty of Science, Technology, Engineering and Mathematics
M303 Further pure mathematics

M303

TMA 04 2020J

Mainly covers Book D *Metric spaces 1*
Cut-off date 2 March 2021

You will find instructions for completing TMAs in the 'Assessment' area of the M303 website. Please remind yourself of these instructions before beginning work on this TMA. You can submit this TMA either by post, together with a completed TMA form (PT3), to arrive by the cut-off date given above, or electronically, as a single PDF file, using the University's online TMA/EMA service.

Half of the TMA is formative, and the other half is summative. The formative questions are designed to either *extend* your understanding of the topics or *introduce* the topics of later books, while the summative questions place more emphasis on *assessing* your understanding of the topics. Both types of question will be marked by your tutor, but (unlike the summative questions) marks for the formative questions do not count towards your final grade. You should submit your solutions to both sets of questions to your tutor by the cut-off date.

The marks allocated to each part of a question are indicated in the margin.

Although many of the questions on this assignment require numerical answers, the questions invariably carry *method marks*.

In phrasing the questions, we have used the words listed below, each of which should be interpreted as indicated.

prove, show, explain, justify	Clear reasoning and explanation for all steps are called for.
determine, find, devise, calculate, compute	An indication of the method used and all working in arriving at an answer should be given.
deduce	Clear explanation of how one result follows from another is required.
solve	Working must be shown. A numerical answer alone is not sufficient.
evaluate, give, write down, list	Answer suffices. No explanation need be given.
hence	No marks will be awarded for any alternative method.

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This TMA is based on Book D (Metric spaces 1), except for the formative Question 8, which covers the start of Book E.

Part A is **summative**. These questions assess your knowledge of the module. You must not discuss these questions in the forums, and your tutor is not allowed to help you directly.

Part B is **formative**. You should choose two questions to submit to your tutor for feedback, and we suggest that one of these is Question 8 since this will help you with the next TMA.

Doing the formative questions will help you to understand key concepts covered in the text. You are encouraged to discuss these questions in the module forums and with your tutor (who is allowed to help you).

Question 8 is designed to help you with TMA 05. Question 9 covers material that is more central to the module, while Questions 10 and 11 are designed to expand your understanding beyond the core material.

Your tutor will mark all of the answers that you submit. But note that only your marks for Part A count towards your overall assessment score.

Part A Summative

Question 1 – 3 marks

This question tests your understanding of sequences and is based on material in Chapters 13 and 14.

(a) Give an example of a sequence in $\mathbb{R}$ that has both an unbounded subsequence and a $d^{(1)}$-convergent subsequence. [*2*]

(b) Briefly explain why any sequence that has the properties given in part (a) must be $d^{(1)}$-divergent. [*1*]

Question 2 – 17 marks

This question tests your understanding of the definition of a novel metric. It asks about material covered in Chapters 14 and 16. A generalisation of this metric is used in Chapter 24 to define a notion of distance between fractals.

Let $X = \{[a, b] : a, b \in \mathbb{R} \text{ and } a \leq b\}$, so X is the set consisting of all possible closed bounded intervals in the real line. Given two intervals $[r, s], [u, v] \in X$, we define

$$\Delta([r, s], [u, v]) = \min\{\varepsilon \geq 0 : [u, v] \subseteq [r - \varepsilon, s + \varepsilon]\}$$

(you may assume that this minimum exists) and use this function to define a distance function on X, $d\colon X \times X \to \mathbb{R}$, by setting

$$d([r, s], [u, v]) = \max\{\Delta([r, s], [u, v]),\ \Delta([u, v], [r, s])\}.$$

(a) Find

$$\Delta([0,1],[0,3]), \quad \Delta([0,3],[0,1]), \quad d([0,1],[0,3]), \quad d([0,0],[2,2]). \qquad [4]$$

(Note that the two intervals in the last example are sets containing single points.)

(b) (i) Prove that d satisfies (M1). [4]

(ii) Prove that d satisfies (M2). [1]

(iii) Suppose that $[u,v]$, $[r,s]$ and $[x,y]$ are in X.

(1) Show that if $\delta_1 = \Delta([u,v],[r,s])$ and $\delta_2 = \Delta([r,s],[x,y])$, then $\Delta([u,v],[x,y]) \leq \delta_1 + \delta_2$.

(2) Show that if $\xi_1 = \Delta([r,s],[u,v])$ and $\xi_2 = \Delta([x,y],[r,s])$, then $\Delta([x,y],[u,v]) \leq \xi_1 + \xi_2$.

(3) Hence, or otherwise, show that (M3) holds for d:

$$d([u,v],[x,y]) \leq d([u,v],[r,s]) + d([r,s],[x,y]). \qquad [5]$$

(c) Let $[a,b] \in X$ and define a sequence $(I_n)_{n\in\mathbb{N}}$ in X by $I_n = [a - 1/n, b + 1/n]$ for $n \in \mathbb{N}$. Show that (I_n) is a d-convergent sequence with limit $[a,b]$. [3]

Question 3 – 9 marks

This question tests your understanding of sequences of functions. It asks about material covered in Chapter 15.

Throughout this question we use the Euclidean metric for $\mathbb{R}$.

For $n \in \mathbb{N}$, define a function $f_n \colon [0,\infty) \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} n\sin(2\pi x), & \text{if } n-1 \leq x \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Sketch, by hand or otherwise, the graphs of f_1, f_2 and f_3. [3]

(b) Determine whether the sequence (f_n) has a pointwise limit. [3]

(c) Determine whether the sequence (f_n) converges uniformly on $[0,\infty)$. [3]

Question 4 – 3 marks

This question tests your understanding of continuity for functions with domain $C[0,1]$. It asks about material covered in Chapter 15.

Show that the function $F \colon C[0,1] \to C[0,1]$ given by $F(f)(t) = \int_0^t x\, f(x)\, dx$ for each $f \in C[0,1]$ and $t \in [0,1]$ is $(d_{\max}, d_{\max})$-continuous on $C[0,1]$. (You may assume that $F(f) \in C[0,1]$.) [3]

Question 5 – 9 marks

This question tests your understanding of material discussed in Chapter 16.

We define three subsets of the plane (using set notation) as follows:

$$A = \{(x, y) \in \mathbb{R}^2 : 2|x| + y < 4\}, \quad B = \{(x, y) \in \mathbb{R}^2 : |y| \leq 2\}$$

and

$$C = A \cap B.$$

(a) Sketch, by hand or otherwise, the set C, taking care to distinguish between points that lie inside and outside the set. [2]

When sketching sets you should follow the conventions used in the module text by using dashed lines to indicate points of the boundary that are not included in the set and solid lines to indicate points that are included. The status of individual points can be indicated by using small circles to indicate points that are not included and small filled-in circles to indicate points that are included.

(b) (i) Determine whether $(2, 0)$ is a boundary point of C for the Euclidean metric. [2]

(ii) Write down in set notation the closure, interior and boundary of C for the Euclidean metric. [3]

(c) Write down the closure of C for the 'mixed' metric d given by

$$d((x_1, x_2), (y_1, y_2)) = d_0(x_1, y_1) + |x_2 - y_2|. \quad [2]$$

(You may assume that d defines a metric for $\mathbb{R}^2$.)

Question 6 – 8 marks

This question tests your understanding of material discussed in Chapter 16.

Let

$$D = \left\{x \in \mathbb{R} : nx^2 - m = 0 \text{ for some } m, n \in \mathbb{N}\right\}.$$

(a) Show that D is countable. [3]

(b) (i) Show that for each $y \in \mathbb{R}$ and each $r > 0$, there is $x \in D$ with $|x - y| < r$. [3]

(ii) Hence, or otherwise, show that D is dense in $\mathbb{R}$ for the Euclidean metric. [2]

Question 7 – 1 mark

This question can be done at any point after Week 14, during your study of Chapters 13–16.

Watch the recording of the *first* informal online lecture on Book D that will be available in Meeting Recordings of the informal online lecture room by the end of Week 14. Write down the date when you watched it. Your tutor will be able to verify that you have viewed the recording. (If you are not able to do this, contact your tutor for advice.) [1]

Part B Formative

Question 8 – 9 marks

In this question you will practise working with ideals in rings that are extensions of degree two over the integers. This question concerns Chapter 17 of Book E, and relates to Question 1 in TMA 05.

Let R be the ring $\mathbb{Z}[\sqrt{-13}]$.

(a) Let A be the subset of R defined by

$$A = \{s + t\sqrt{-13} : s, t \in \mathbb{Z},\ s - t \equiv 0 \,(\mathrm{mod}\ 7))\}.$$

Show that A is an ideal in the ring R. [3]

(b) The subset

$$B = \{u + v\sqrt{-13} : u, v \in \mathbb{Z},\ u + v \equiv 0 \,(\mathrm{mod}\ 7))\}$$

is also an ideal in the ring R. Show that $AB = \langle 7 \rangle$. [6]

Question 9 – 9 marks

In this question we ask you to sketch particular balls and spheres for various metrics on the plane. You should be able to attempt this question after reading Chapter 14.

Sketch, by hand or otherwise, the sets

$$B_d((0,0),1), \quad B_d[(0,0),1] \quad \text{and} \quad S_d((0,0),1)$$

for the metric d on the plane when d is given by each of the following, for $(a_1, a_2), (b_1, b_2) \in \mathbb{R}^2$. (You may assume that each of these defines a metric.)

(a) $d((a_1,a_2),(b_1,b_2)) = 6|b_1 - a_1| + |b_2 - a_2|$ [3]

(b) $d((a_1,a_2),(b_1,b_2)) = (|b_1 - a_1|^4 + |b_2 - a_2|^4)^{1/4}$ [3]

(c) $d((a_1,a_2),(b_1,b_2)) = d_0(a_1,b_1) + 6|b_2 - a_2|$ [3]

Question 10 – 16 marks

In this question we introduce a metric for the rational numbers, known as the 5-adic metric*, that is very different to the usual one for $\mathbb{Q}$ that is induced by the Euclidean metric for $\mathbb{R}$. You should be able to do this question after reading Chapters 14–15.*

Using results from Book A, one can show that for each non-zero rational number x, there is a unique $m \in \mathbb{Z}$ such that

$$x = 5^m \frac{a}{b},$$

where $a \in \mathbb{Z} - \{0\}$, $b \in \mathbb{N}$, and 5 does not divide either of a and b. We use this fact to define a function $f_5 \colon \mathbb{Q} \to \mathbb{R}$ by

$$f_5(x) = \begin{cases} 5^{-m}, & \text{if } x = 5^m \dfrac{a}{b} \text{ and 5 does not divide either of } a \text{ and } b, \\ 0, & \text{if } x = 0. \end{cases}$$

Now define a distance function $d_5 \colon \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ by

$$d_5(r, s) = f_5(r - s).$$

(a) Write down $f_5(1)$, $f_5(5)$ and $f_5(5^n)$ for $n \in \mathbb{N}$. [2]

(b) Show that f_5 has the following properties.

(i) For each $x \in \mathbb{Q}$, $f_5(x) \geq 0$, with equality if, and only if, $x = 0$. [2]

(ii) For each $x \in \mathbb{Q}$, $f_5(-x) = f_5(x)$. [2]

(iii) For $x, y \in \mathbb{Q}$, $f_5(x + y) \leq \max\{f_5(x), f_5(y)\}$. [6]

(c) Using the results of part (b), or otherwise, show that d_5 is a metric for $\mathbb{Q}$. [4]

Question 11 – 16 marks

In this question we define a metric on a group. The question revises some techniques from Book B in addition to enabling you to practise some techniques from Book D.

Let G be a group, let $n \in \mathbb{N}$, and suppose that $H_0, H_1, H_2, \ldots, H_n$ are subgroups of G such that

$$\{e\} = H_0 \leq H_1 \leq \cdots \leq H_n = G, \quad \text{with } H_i \neq H_{i-1} \text{ for } i = 1, 2, \ldots, n.$$

We define a distance function $d_G \colon G \times G \to \mathbb{R}$ as follows:

$$d_G(g, h) = k, \quad \text{where } gh^{-1} \in H_k \text{ and } gh^{-1} \notin H_i \text{ for all } i < k.$$

(a) Prove that d_G is a metric for G by showing the following.

(i) d_G satisfies (M1). [2]

(ii) d_G satisfies (M2). [3]

(iii) d_G satisfies (M3). [6]

(b) Show that a sequence (g_i) of elements in G is d_G-convergent if, and only if, the sequence (g_i) is eventually constant. [5]